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ABSTRACT

Part I of this report describes a method for tidal analysis based on high and low water observations and also discusses the results of various test runs. Part II gives detailed information on certain aspects of the calculations and Part III is a user's manual for the relevant computer program.

Wherever possible, terminology, computations and computer input and output formats are in accord with two earlier reports (Foreman, 1977, 1978) dealing with equally spaced tidal height and current data.

Users who wish to receive updates of the computer programs should send their names and addresses, and type of computer used, to the authors.

PART I: GENERAL DESCRIPTION

1 INTRODUCTION

It is customary nowadays, in tidal heights analysis, to use hourly values of surface elevation obtained either from a digital tide gauge or by sampling a continuous record of water level. However, there is a mass of historical tidal data in the form of times and magnitudes of successive high and low water levels, and even today there are situations in which it is convenient to record tidal elevations in this form. The fact that the time intervals between successive tidal extrema vary considerably means that most standard computer programs (e.g. Foreman, 1977) used for tidal (harmonic) analysis of conventional hourly-sampled data are not applicable. Yet these are generally based on least squares fitting of tidal-frequency constituents to the data and there is no reason, in principle, why irregularly-spaced data cannot be analysed in the same way. In such an approach to analysis of high and low water levels which, for convenience, we term "high–low analysis", there is an upper limit to the tidal constituent frequencies which can be included. Basic sampling theory shows that constituents which are sampled fewer than two times per cycle become irretrievably confused with lower frequency constituents — a type of error known as "aliasing". Knowing the times and magnitudes of low and high waters should be somewhat better than simply having two elevation samples per tidal cycle, since the time derivative of surface elevation is zero at extrema. Assuming that, on average, four extrema occur per (lunar) day, it may be possible, using all the information implicit in the extremal amplitudes and times, to resolve constituents with periods as low as six hours. A conservative position is taken in the test cases described later in Section 3, where no attempt is made to determine constituents above semidiurnal frequency. It is shown, however, that failure to use the derivative information leads to less accurate estimation of semidiurnal constituents — a result probably attributable to aliasing of terdiurnal and quarter-diurnal constituents. This effect will be more noticeable in locations where higher-frequency constituents are relatively large, for instance, in shallow water.

Although high and low water observations have been taken over many months at a few locations, a record length of one month was taken in the numerical tests as being more typical. Consequently, low-frequency constituents (e.g. fortnightly, monthly) were omitted from the analysis, since they often show considerable variation from month to month due to meteorological effects; there is no theoretical obstacle to their inclusion, however.

Visual observations of tidal extrema often cover daylight hours only, in which case either one or two observations per day are missing and, at some sites, no observations at all were made on Sundays. These defects affect the accuracy of high–low analysis, and use of longer records is the obvious countermeasure. The missing data points cause no practical difficulty in least squares fitting procedures originally set up for unevenly spaced data.

A most important consideration in analyses based on high and low water data is the reliability of the observed times. Ideally, observations of water level should be made from some time before the extreme value until some time afterward, and then plotted so that the peak value and its time of occurrence can be estimated accurately. But, even so, short-period waves introduce error into all the observations and it is reasonable to expect a minimum random error in times of high and low water of the order of a few minutes; the effect of such random timing errors is discussed later. If the correct observation procedure, outlined above, is not followed, the times of extrema made by most observers tend to be several minutes late. Consistent errors of this type affect the phases of constituents by a calculable amount, since the period of each constituent is known, and corrections are possible if some estimate of the timing error can be made.

Previous work on tidal analysis of high and low water observations ranges from the traditional sequential methods discussed by Schureman (1958), which yield only moderately accurate estimates of the principle constituents, to least squares analysis (e.g. Zetler, Schuldt, Whipple and Hicks, 1965) essentially similar to that used here. The purpose of the present report is to combine an efficient least squares algorithm with the full suite of nodal modulation, astronomical argument correction and full inference calculations, while maintaining maximum compatibility of terminology and format with other tidal analysis programs currently in use at the Institute of Ocean Sciences.

2 METHOD OF ANALYSIS

2.1 Choice of Constituents

The magnitude of tidal constituents usually does not vary rapidly with latitude or longitude and, if no prior information is available from the site in question, it is fairly safe to assume that the same constituents will dominate the tidal behaviour at the new location as at other known stations in the same geographical area. In the numerical tests below, selection of first magnitude constituents was simple, since conventional harmonic analysis results for the test site were already available. The diurnals O_1 and K_1 and semidiurnals N_2, M_2 and S_2 were clearly, most significant; in certain tests, the diurnal P_1 and semidiurnal K_2 were also included, being inferred from K_1 and S_2, respectively (see Section 5.3). Constituents outside the diurnal to semidiurnal frequency range were excluded for reasons discussed in Section 1. A constant term, Z_0, was included as a matter of course, since surface elevations are seldom measured directly relative to mean water level. It may be noted that, with a record length of one month, the system of equations to be solved is substantially overdetermined and there is no difficulty in adding minor constituents if desired.

2.2 Least Squares Fitting of Sinusoidal Constituents to High and Low Water Data

Assuming that a sequence of high and low water observations, y_i, and the corresponding times, t_i, for $i = 1, \ldots, N$ at which they occurred, are given, we wish to find a function

$$y(t) = A_0 + \sum_{j=1}^{M} A_j \cos 2\pi(\sigma_j t - \phi_j), \tag{1}$$

in which the constituent frequencies, σ_j, and the number of constituents, M, are specified beforehand, but the amplitudes, A_j, and phases, ϕ_j, remain to be chosen so that the values, $y(t_i)$, of the fitting function at the sampling instants, t_i, agree as well as possible with the contemporaneous observed elevations, y_i, i.e.

$$y_i - \left[A_0 + \sum_{j=1}^{M} A_j \cos 2\pi(\sigma_j t_i - \phi_j)\right] = \epsilon_i \simeq 0, \qquad i = 1, \ldots, N. \tag{2}$$

Further, at the observation times, t_i, the time derivative, $y'(t)$, of the fitting function should be approximately zero, i.e.

$$y'(t_i) = -\sum_{j=1}^{M} 2\pi\sigma_j A_j \sin 2\pi(\sigma_j t_i - \phi_j) = \delta_i \simeq 0. \tag{3}$$

The fitting errors, $\epsilon_i\,\delta_i$, cannot be reduced exactly to zero when the number of arbitrary constants $(2M+1)$ in the expression for $y(t)$ is less than $2N$, the number of equations (2) and (3) to be satisfied. A commonly adopted compromise in such overdetermined problems is to minimize the sum of the squares of errors at the observation times, which means, in the present case, choosing the A_j and ϕ_j so as to minimize the error function

$$E = \sum_{i=1}^{N} \Big\{ [y_i - y(t_i)]^2 + [wy'(t_i)]^2 \Big\} = \sum_{i=1}^{n} \Big\{ \epsilon_i^2 + w^2\delta_i^2 \Big\}, \tag{4}$$

i.e. to find a *least squares fit* to the available data. The inclusion of an arbitrary positive weighting coefficient, w, in (4) permits control of the emphasis to be placed on satisfying the zero derivative condition compared to that placed on having $y(t)$ fit the observed elevations accurately. For instance, $w = 1.0$ indicates that equal emphasis is given to both conditions, whereas $w = 0$ means that the requirement that $y'(t)$ should be approximately zero at each t_i is simply ignored.

Details of the algorithm used for numerical minimization of E in (4) are given in Section 4.

3 NUMERICAL TESTS

To test the effectiveness and accuracy of high–low analysis by least squares fit, some numerical experiments were carried out on surface level observations collected in 1974 at Prince Rupert, British Columbia, a fairly typical West Coast port with relatively deep approaches, where the tide is predominantly semidiurnal with significant diurnal contributions. In order to have some basis for judging the high–low analysis, four conventional harmonic analyses of hourly heights were carried out first.

3.1 Harmonic Analyses

Analysis 1

A full 12-month hourly height harmonic analysis (Foreman, 1977) for 1974 is listed in Appendix 8.1. The constant component plus the major diurnal and semi-diurnal constituents from this 68-constituent analysis form the first row of Table 1. That there is little non-tidal contribution to water level variation at Prince Rupert is evident from the fact that the residual elevation, after removal of tidal constituents found in Analysis 1, had an rms value of 0.13 m, which is approximately 2% of the tidal range.

Since seasonal variation at Prince Rupert is very slight and one-month analyses were the principal topic of interest, all subsequent tests were confined to a single arbitrarily-selected month — January 1974.

Table 1 Computed Amplitudes (m) and Phases (deg) of Principal Constituents.

Analysis Number	Analysis details	Number of constituents*	Z_0	O_1		P_1		K_1		N_2		ν_2		M_2		S_2		K_2	
			Amp.	Amp.	Ph.	Amp.	Ph.	Amp.	Ph.	Amp.	Ph.	Amp.	Ph.	Amp.	Ph.	Amp.	Ph.	Amp.	Ph.
H.A. 1	12 months (Jan.–Dec. 1974)	68	3.8714	0.3125	132.5	0.1606	135.8	0.5144	139.5	0.3952	14.9	0.0766	16.5	1.9564	35.8	0.6446	59.3	0.1738	50.7
H.A. 2	31 days (Jan. 1974), no inference	36	3.8972	0.3154	132.8	—	—	0.6489	152.7	0.4699	12.4	—	—	1.9358	35.5	0.6644	72.9	—	—
H.A. 3	31 days, with inference	40	3.8972	0.3154	132.8	0.1700	137.2	0.5444	140.9	0.3972	14.8	0.0770	16.4	1.9358	35.5	0.6791	58.1	0.1831	49.5
H.A. 4	31 days, with inference	9	3.8971	0.3168	131.4	0.1681	137.5	0.5386	141.1	0.4035	15.2	0.0782	16.8	1.9321	35.3	0.6739	57.4	0.1817	48.8
H-L. 5	31 days, $w = 1$, no inference	6	3.8873	0.3133	130.4	—	—	0.6405	152.7	0.4176	12.2	—	—	1.9497	35.8	0.6877	71.6	—	—
H-L. 6	31 days, $w = 1$, with inference	9	3.8873	0.3133	130.4	0.1677	137.3	0.5374	140.9	0.3532	14.6	0.0685	16.2	1.9497	35.8	0.7028	56.8	0.1895	48.1
H-L. 7	31 days, $w = 0$, with inference	9	3.8868	0.3095	131.8	0.1664	139.1	0.5330	142.8	0.3037	19.2	0.0589	20.9	1.9725	39.1	0.7062	59.9	0.1904	51.3
H-L. 8	31 days with gaps, $w = 1$, with inference	9	3.8531	0.3146	130.2	0.1665	142.9	0.5332	146.5	0.3470	15.0	0.0673	16.6	1.9572	35.6	0.6793	56.8	0.1831	48.2
H-L. 9	31 days, timing errors, $w = 1$, with inference	9	3.8874	0.3183	130.0	0.1684	137.6	0.5393	141.3	0.3580	15.2	0.0694	16.9	1.9533	35.8	0.6981	56.6	0.1882	48.0
H-L. 10	31 days, timing errors, $w = 1$, with inference	9	3.8876	0.3127	131.9	0.1656	139.4	0.5305	143.1	0.3058	20.1	0.0593	21.8	1.9828	39.4	0.7048	59.7	0.1900	51.1

* including mean (constant term)

Analysis 2

Hourly heights for January 1974 were analysed for the suite of 36 constituents which can be resolved from a 31-day record using a Rayleigh (resolution) criterion value of 0.97 (Foreman, 1977, p. 9). Constituents, P_1, ρ_1, ν_2 and K_2, which can only be obtained by inference in a one-month analysis (see Section 5.3), were omitted.

Analysis 3

Though otherwise similar to Analysis 2 above, this case used 40 constituents, P_1, ρ_1, ν_2 and K_2 being inferred from K_1, Q_1, N_2 and S_2, using inference constants calculated from the one-year Analysis 1.

Comparison of Analyses 1, 2 and 3 shows clearly how much more accurately the constituents, K_1, N_2 and S_2, can be estimated from monthly records when inference is used. Of course, in this example, the inference constants are optimum since they were calculated from a longer record at the same site; in practice, inference constants may have to be estimated from data at a neighbouring site which can result in a less marked improvement.

Since tidal constituent frequencies are not harmonics of a single fundamental frequency, the results obtained by least squares fit analysis depend on the number of constituents included. The effect on the major constituents of omitting many of the minor constituents can be seen on comparing the 40-constituent Analysis 3 with the following:

Analysis 4

This test is a monthly harmonic analysis identical to Analysis 3 but for the fact that only five major, plus three inferred constituents and a constant term are included in the least squares fit. These are the constituents included in most of the high–low analyses, described below.

3.2 High–Low Analyses

The heights and times of high and low waters used in the following tests were taken from a strip-chart record. The times are estimated to lie within three minutes of actual high or low water; heights are correct to within ± 1.5 cm (0.05 ft). As explained in Section 2.2, the relative importance accorded to the fact that the time derivative of elevation is zero at high and low water, is reflected in the magnitude of the weighting coefficient in the least squares fit procedure.

Analysis 5

This is a least squares fit of five major constituents to 119 consecutive high and low waters (times and magnitudes) at Prince Rupert in January 1974. No inference of P_1, ν_2 and K_2 was carried out. The zero derivative information was accorded the same significance as the elevation readings, i.e. the weighting coefficient, w, was taken as unity.

Comparing Analysis 5 with 2, it is clear that high–low analysis is capable of determining constant, diurnal and semidiurnal constituents with very satisfactory accuracy. That inference is a useful additional feature in high–low analysis is verified in the following test:

Analysis 6

This high–low analysis of the data already examined in Analysis 5 employs the same five major constituents, but also infers P_1, ν_2 and K_2, using the same inference constants as in Analyses 3 and 4.

Comparing these results with Analysis 5 and referring also to the constituents found in the one-year Analysis 1, one finds the same degree of improvement in constituents K_1, N_2 and S_2 as when inference was introduced into the harmonic analysis (Analysis 3 versus Analysis 2). It can be concluded that inference is worthwhile in high–low analysis whenever reasonably reliable inference constants are available.

Analysis 7

This test is identical to Analysis 6, except that the zero derivative information is not used (zero weighting coefficient). Inference was applied as in the foregoing test.

The results of Analysis 7 obviously differ from Analysis 3 rather more than Analysis 6 does. In other words, omission of derivative information impairs high–low analysis somewhat. Nevertheless, Analysis 7 is close enough to Analysis 3 to be considered quite satisfactory. This is an important conclusion, since Analysis 7 is probably representative of the results which can be expected from irregularly-spaced sets of elevation observations which are not necessarily extrema. In fact, the computer program used for high–low analyses can be used unchanged for any arbitrarily-chosen set of observations, provided the derivative weighting coefficient, w, is zero.

3.2.1 High–low sequences with gaps

High and low waters which occurred during hours of darkness are often missing from records taken visually. In some cases, no readings were made on Sundays. In order to find out the effects of such gaps on the accuracy of high–low analyses, the data already used in Analyses 5 through 7 were modified by deleting observations made between 9 p.m. and 5 a.m. (local time) on weekdays and Saturdays, and all day Sunday. This reduced the total number of observations for January 1974 from 119 to 73.

Analysis 8

The reduced set of observations described above was subjected to high–low analysis for five major and three inferred constituents, with unit weighting for the derivative information.

The results in Table 1 agree slightly less well with Analysis 3 than the various earlier analyses based on 119 observations but, nevertheless, are remarkably close.

3.2.2 High–low analysis with timing errors

In order to simulate the effects of errors in timing when high and low water are estimated purely by eye, the genuine data already used in Analyses 5 to 8 were altered by adding randomly-generated timing errors, uniformly distributed in the range -15 to $+15$ min, to the individual observation times. The magnitudes of the observed high and low waters were not altered.

Analysis 9

A high–low analysis of the data with simulated timing errors, as described above, was carried out using inference and with a weighting coefficient of 1.0 for the zero derivative condition.

Analysis 10

The preceding analysis was repeated with a weighting coefficient of zero, i.e. the fact that the observed elevations were known to be maxima or minima was ignored.

The results of Analyses 9 and 10 show that the former gives estimates closer to the true tidal content of the data as defined by Analyses 3 and 4.

3.3 Conclusions

It is clear from the numerical tests in Section 3 that high–low analysis consisting of a least squares fit of major diurnal and semidiurnal tidal constituents to high and low water levels can yield very satisfactory estimates of amplitudes and phases of the constituents involved, at least for records about one month in length. It is expected that minor constituents of semidiurnal and lower frequencies can be resolved where longer records are available. However, higher frequency constituents are best considered as noise, since even taking the derivative information into account, the upper limit placed by sampling theory on the resolvable frequencies is somewhere between semidiurnal and quarter-diurnal.

The tests analyses indicate that:

(i) it is always best to use, rather than to ignore, the information that the observations of elevation are also extrema,

(ii) the estimates of some major constituents are improved by *inferring* certain subsidiary constituents,

(iii) a lack of night-time observations and occasional missing days impair high–low analysis only slightly, and

(iv) high–low analysis is fairly insensitive to errors of several minutes in the observed times of high or low water.

PART II: DETAILS OF PROCEDURES

4 LEAST SQUARES FIT WITH MODIFIED GRAM–SCHMIDT ALGORITHM

The original system of overdetermined equations which gives rise to the least squares fit problem is

$$\left.\begin{aligned} y(t_i) - y_i = A_0 + \sum_{j=1}^{M} A_j \cos 2\pi(\sigma_j t_i - \phi_j) - y_i = 0 \\ y'(t_i) = -\sum_{j=1}^{M} 2\pi\sigma_j A_j \sin 2\pi(\sigma_j t_i - \phi_j) = 0 \end{aligned}\right\} i = 1, \ldots, N \tag{5}$$

in the notation of Section 2. It is convenient to change variables to $C_0 = A_0$, $C_j = A_j \cos 2\pi\phi_j$, $S_j = A_j \sin 2\pi\phi_j$ for $j = 1, \ldots, M$, since the above equations then become

$$\begin{aligned} C_0 + \sum_{j=1}^{M} (C_j \cos 2\pi\sigma_j t_i + S_j \sin 2\pi\sigma_j t_i) = y_i \\ \sum_{j=1}^{M} 2\pi\sigma_j (C_j \sin 2\pi\sigma_j t_i - S_j \cos 2\pi\sigma_j t_i) = 0 \end{aligned} \tag{6}$$

which are linear in the new unknowns, C_0, C_j, S_j. The A_j and ϕ_j can be recovered later from the C_j, S_j by means of the formulae

$$\begin{aligned} A_j &= (C_j^2 + S_j^2)^{1/2} \\ 2\pi\phi_j &= \tan^{-1} \frac{S_j}{C_j}. \end{aligned} \tag{7}$$

We now review the reasons for preferring the Modified Gram–Schmidt least squares fit algorithm to that used in the harmonic analysis of hourly heights (Foreman, 1977). Given an overdetermined system of equations written in the matrix form $A\mathbf{x} = \mathbf{b}$, a common approach to the linear least squares fit problem is to form and solve the normal equations $A^T A\mathbf{x} = A^T\mathbf{b}$. These are not overdetermined, but are frequently ill-conditioned, making the solutions very sensitive to round-off errors, etc. In order to preserve accuracy, it is preferable to compute the least squares solution directly from the original overdetermined equations by orthogonalization procedures, such as Householder triangularization, singular value decomposition or the Modified Gram–Schmidt method. For instance, solution by normal equations requires double precision arithmetic to give the same accuracy as Householder's method achieves in single precision (Barrodale and Erikson, 1978).

Nevertheless, when the number of equations, N, is much larger than the number of parameters, M, the normal equations approach has its advantages. Not only do the formation and solution of these equations require about half as many operations as an orthogonalization technique, but if these equations can be formed directly from the data rather than from the overdetermined system, then only $M(M+1)/2$ storage locations, as opposed to at least NM, are required. This is, in fact, why the normal equations were used in the harmonic tidal heights analysis of hourly observations (Foreman, 1977). There, the normal equations are formed directly and efficiently (the use of trigonometric identities avoids rounding errors usually encountered with cumulative sums) and aggravation of the problem's already ill-conditioned nature is avoided. The storage savings can be significant — for instance, in a one-year analysis where the approximate values for N and M are 8760 and 137 respectively. In fact, on some installations there may not be sufficient storage for an overdetermined array of these dimensions.

Since the numbers of observations and constituents will generally be smaller when high–low analysis is used, storage considerations will be less important. Also, the identities used to form the normal equations in the regularly sampled case no longer apply. Orthogonalization methods are, therefore, preferable and the Modified Gram–Schmidt algorithm (Barrodale and Stuart, 1974) was selected, since it is competitive in all respects with the Householder method and was already available on the Institute of Ocean Sciences' computer.

Orthogonalization methods obtain the least squares solution to the matrix equation $A\mathbf{x} = \mathbf{b}$ by forming an equivalent system of equations which is easier to solve. In particular, the classical Gram–Schmidt technique does this by calculating an orthogonal set[1] of vectors $\{\mathbf{q}_1, \ldots, \mathbf{q}_{n+1}\}$ such that for $k = 1, \ldots, n+1$, the set $\{\mathbf{q}_1, \ldots, \mathbf{q}_k\}$ spans the same k-dimensional subspace as the given set of linearly independent vectors $\{\mathbf{a}_1, \ldots, \mathbf{a}_k\}$, where the set $\{\mathbf{a}_1, \ldots, \mathbf{a}_{n+1}\}$ are the columns of the augmented matrix $A : \mathbf{b}$ arising from the overdetermined system $A\mathbf{x} = \mathbf{b}$. The set of mathematical formulae which calculate the q_j vectors iteratively are as follows:

$$\mathbf{q}_1 = \mathbf{a}_1, \tag{8}$$

$$\mathbf{q}_j = \mathbf{a}_j - \sum_{i=1}^{j-1} r_{ij}\mathbf{q}_i \qquad j = 2, \ldots, n, \tag{9}$$

where

$$r_{ij} = \frac{\mathbf{a}_j^T \mathbf{q}_i}{\mathbf{q}_i^T \mathbf{q}_i}. \tag{10}$$

In order to convert these equations to matrix notation, let A be the matrix with columns, $\mathbf{a}_j$; Q be the matrix with columns, $\mathbf{q}_j$, and R be the upper triangular matrix with unit diagonal elements and super-diagonal elements given by (10). Then equations (8) through (10) can be written as $A = QR$.

The Modified Gram–Schmidt method is a variation of the classical technique which makes use of the fact that the value of the inner product, $\mathbf{a}_j^T \mathbf{q}_i$, in equation (10) will not change if $\mathbf{a}_j$ is replaced by any vector of the form

$$\mathbf{a}_j^{(i)} = \mathbf{a}_j - \sum_{k=1}^{i-1} \alpha_k \mathbf{q}_k, \tag{11}$$

where the α_k are a set of arbitrary numbers. In particular, if one chooses the numbers, α_k, so as to minimize the norm of the vector, $\mathbf{a}_j^{(i)}$, it can be shown that replacing $\mathbf{a}_j$ by $\mathbf{a}_j^{(i)}$ in the

[1] the set of vectors $\{\mathbf{q}_1, \ldots, \mathbf{q}_n\}$ is said to be orthogonal if $\mathbf{q}_1^T \mathbf{q}_j = 0$ for all $i \neq j$. If, in addition, $\mathbf{q}_i^T \mathbf{q}_i = 1$, the set is orthonormal.

classical Gram–Schmidt orthogonalization produces a method that is more stable numerically. The Modified Gram–Schmidt (MGS) algorithm, is described by the following equations (Lawson and Hanson, 1974):

$$\mathbf{a}_j^{(i)} = \mathbf{a}_j \qquad j = 1, \ldots, n, \tag{12}$$

$$\left.\begin{aligned} \mathbf{q}_i &= \mathbf{a}_i^{(i)}, && (13)\\ d_i^2 &= \mathbf{q}_i^T \mathbf{q}_i, && (14)\\ &\left.\begin{aligned} r_{ij} &= \frac{\mathbf{a}_j^{(i)^T} \mathbf{q}_i}{d_i^2} && (15)\\ \mathbf{a}_j^{(i+1)} &= \mathbf{a}_j^{(i)} - r_{ij} \mathbf{q}_i. && (16) \end{aligned}\right\} \; j = 1+1, \ldots, n, \end{aligned}\right\} \; i = 1, \ldots, n,$$

In order to use the MGS orthogonalization to minimize the sum of the squares of the residuals (i.e. $\|A\mathbf{x} - \mathbf{b}\|^2$) for the overdetermined system, $A\mathbf{x} = \mathbf{b}$, first form the augmented $m \times (n+1)$ matrix $A' = [A : \mathbf{b}]$. A' is then orthogonalized to obtain

$$A' = Q'R', \tag{17}$$

where the column vectors given by (13) constitute the $m \times (n+1)$ matrix, Q', and R' is an upper triangular matrix with unit diagonal elements and super-diagonal elements given by (15). Defining D' to be the $(n+1) \times (n+1)$ diagonal matrix with elements specified by (14), a new $m \times m$ orthogonal matrix,[2] Q_0, is introduced such that

$$Q_0 \begin{bmatrix} D' \\ 0 \end{bmatrix} = Q',$$

where 0 has dimension $(m-n+1) \times (n+1)$. This means that the first $n+1$ columns of Q_0 will be $\mathbf{q}_i/d_i^2$ for $i = 1, \ldots, n+1$ and the remaining $m-n+1$ need only complete the orthonormal set. Partitioning R' and D' into $\begin{bmatrix} R & \mathbf{c} \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} D & \mathbf{0} \\ 0 & d_{n+1} \end{bmatrix}$ respectively, where both D and R are $n \times n$, we can then write

$$A' = Q_0 \begin{bmatrix} D' \\ 0 \end{bmatrix} R' = Q_0 \begin{bmatrix} DR & Dc \\ 0 & d_{n+1} \\ 0 & 0 \end{bmatrix}.$$

Making use of the property that $\|B^T x\| = \|x\|$ for any orthogonal matrix, B, it then follows that

$$\begin{aligned} \|A\mathbf{x} - \mathbf{b}\|^2 &= \|Q_0^T (A\mathbf{x} - \mathbf{b})\|^2 \\ &= \|D(R\mathbf{x} - \mathbf{c})\|^2 + d_{n+1}^2. \end{aligned}$$

Therefore, the minimum value of $\|A\mathbf{x} - \mathbf{b}\|^2$ is d_{n+1}^2 and is attained by the vector, $\mathbf{x}_0$, which satisfied $R\mathbf{x} = \mathbf{c}$.

[2] An orthogonal matrix, Q, satisfies the condition, $Q^T = Q^{-1}$. As a result, its columns form an orthonormal set.

A later version of MGS (Barrodale and Stuart, 1974) is more efficient, in that the matrix equation, $R\mathbf{x} = \mathbf{c}$, is solved as part of the orthogonalization process. Specifically, A' is now defined as the larger partitioned matrix $A' = \begin{bmatrix} A & \mathbf{b} \\ I & 0 \end{bmatrix}$, where I is the $n \times n$ identity matrix and $\mathbf{0}$ is the $n \times 1$ vector of zeros. Applying MGS to A' results in the following matrix of orthogonal columns $\begin{bmatrix} Q & \mathbf{r} \\ R^{-1} & -R^{-1}\mathbf{c} \end{bmatrix}$ where $[Q\ \mathbf{r}]$ is the Q' of equation (17) and $R, \mathbf{c}$ are the same as in the partition of R'. Thus, the least squares solution, $\mathbf{x}_0 = R^{-1}\mathbf{c}$, can be easily removed from this matrix.

Moreover, since

$$A' = \begin{bmatrix} A & \mathbf{b} \\ I & 0 \end{bmatrix} = \begin{bmatrix} Q & \mathbf{r} \\ R^{-1} & -R^{-1}\mathbf{c} \end{bmatrix} \begin{bmatrix} R & \mathbf{c} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} QR & Q\mathbf{c} + \mathbf{r} \\ I & 0 \end{bmatrix},$$

it follows that $Q = AR^{-1}$, $\mathbf{b} = AR^{-1}\mathbf{c} + \mathbf{r}$ and hence $\mathbf{r} = \mathbf{b} - A\mathbf{x}_0$ is the vector of residuals corresponding to the least squares solution of the overdetermined system.

5 MODIFICATIONS TO RESULTS OF LEAST SQUARES ANAYLSIS

5.1 Nodal Modulation

The tidal potential contains many more sinusoidal constituents than are commonly sought in tidal analysis. Due to the small frequency differences between many of these (and the great length of record required for their separation) and the relatively small expected amplitude of some, it is not feasible to analyse for all of them. The standard approach is to lump together constituents which have the same first three Doodson numbers (see Godin, 1972) and to assume that each such cluster can be replaced by a single sinusoid having the same frequency as the major constituent (in terms of tidal potential amplitude) in the cluster. This major contributor lends its name to the cluster and lesser constituents are termed its satellites. An amplitude and phase are then calculated from the data for each apparent major constituent (in fact, for the replacement sinusoid representing each cluster). However, since these values represent the cumulative effect of all constituents in the cluster and the latter all differ slightly in frequency, the amplitude and phase of the replacement sinusoid vary slowly in time and do not provide a basis for predicting the contribution of the cluster to the tidal signal at a subsequent time. To avoid this difficulty, the time-invariant amplitude and phase of the major constituent in each cluster are calculated from those of the replacement sinusoid. This adjustment procedure is known as “nodal modulation”. To predict the contribution of the cluster at a later time, the major constituent is first calculated and then the nodal modulation corrections to its amplitude and phase are applied in the reverse sense to obtain the contribution of the cluster as a whole.

In this report, the replacement sinusoid for the jth cluster is first written as

$$A_j \sin 2\pi[\sigma_j t_1 - \phi_j], \tag{18}$$

where A_j and ϕ_j are termed the raw amplitude and raw local phase respectively. Time, t_1, is measured from the midpoint of the record being analysed. A customary notation for (18) is

$$f_j(t) a_j \cos 2\pi[\sigma_j t_1 - \theta_j + u_j(t)], \tag{19}$$

which expresses the relation between the cluster contribution in terms of the amplitude, a_j, and phase, θ_j, of its major constituent. The nodal modulation terms of $f_j(t)$ and $u_j(t)$ vary slowly with time and for records up to one year in length very little error is introduced by assuming them to be constant and equal to their value at $t_1 = 0$, the midpoint of the record. Further details of the nodal modulation calculations and the evaluation of $f_j(t)$ and $u_j(t)$ are given in Foreman (1977, p. 24).

5.2 Astronomical Argument Corrections

The astronomical argument correction arises from the need to express all constituent phase lags with respect to a universal time and space origin. Instead of regarding each tidal constituent as the result of a particular component in the tidal potential, an artificial causal agent can be attributed to each constituent in the form of a fictitious star which travels around the equator with angular speed equal to that of its corresponding constituent. Making use of this conceptual aid, the astronomical argument of a given tidal constituent can be viewed as the angular position (longitude) of its fictitious star. For historical reasons, all such arguments or longitudes are expressed relative to the Greenwich meridian and can, consequently, be expressed as functions of time only. The replacement sinusoid (19) is often written as

$$f_j(t)a_j \cos 2\pi[V_j(t) + u_j(t) - g_j], \tag{20}$$

where $V_j(t)$ is the longitude of the fictitious star relative to Greenwich and the new time variable, t, has an absolute datum such as some calendrical landmark. The term, g_j, is the "Greenwich phase lag" of the jth constituent.

Ignoring the minute effects of long-term changes in the astronomical variables (see Foreman, 1977, p. 8),

$$V_j(t) = \sigma_j t + V_{0j}$$

where V_{0j} is a phase correction due to the change of time datum. If t_c is the midpoint of the record on the new time scale, t, then $t = t_1 + t_c$ and

$$\begin{aligned} A_j \cos 2\pi[\sigma_j t_1 - \phi_j] &= A_j \cos 2\pi[\sigma_j t - \sigma_j t_c - \phi_j] \\ &= A_j \cos 2\pi[V_j(t) - V_{0j} - \sigma_j t_c - \phi_j] \\ &= A_j \cos 2\pi[V_j(t) - V_j(t_c) - \phi_j]. \end{aligned}$$

Comparing this with (20), we see that the Greenwich phase lag is, in fact

$$g_j = \phi_j + V_j(t_c) + u_j(t_c).$$

5.3 Inference

Inference is the term used in tidal analysis to describe the extraction of certain important constituents excluded at the least squares fit stage on the grounds of insufficient record length but deduced afterwards from included constituents to which they bear a known amplitude and phase relationship. When accurate inference constants (amplitude ratio and phase difference) are available, inference not only yields amplitudes and phases for the inferred constituents, but also significantly reduces periodic variations in the estimated amplitudes and phases of the reference constituents. The computational steps involved in inference are given in detail in Foreman (1977). That material will not be repeated here, but should be read in the light of the following comments.

The question of when constituents should be included directly in the least squares analysis, and when they should be inferred, is not easily answered. The Rayleigh criterion, which is used to select constituents in the harmonic analysis of hourly tidal heights (see Foreman, 1977, p. 9), is incomplete in its presumption that a record of length, T, is required to distinguish constituents with a frequency separation of T^{-1}. In fact, it conflicts with the algebraic viewpoint whereby, for any four independent observations, one can obtain four equations and solve for four unknowns (two amplitudes and two phases), regardless of the frequency separation. The missing consideration in both of these viewpoints is that sea-level observations contain, in addition to discrete tidal signals, contributions from a continuous noise spectrum of geophysical origin and from random errors in recording the observations. Taking these effects into account, Munk and Hasselman (1964) showed that meaningful information can be gained about the frequencies, σ_1 and σ_2, provided that

$$|\sigma_2 - \sigma_1| > \frac{T^{-1}}{(\text{signal/noise level})^{1/2}}.$$

It is interesting that essentially the same result can be derived by considering the sensitivity of solutions of a linear system to the condition number of its coefficient matrix. The following account is based on the detailed discussion in Ortega (1972). If $K(A) = \|A\| \, \|A^{-1}\|$ is the condition number[3] of matrix A and $\mathbf{x}$, $\hat{\mathbf{x}}$ are such that $A\mathbf{x} = \mathbf{b}$ and $A\hat{\mathbf{x}} = \mathbf{b} + \boldsymbol{\delta}\mathbf{b}$, then

$$\frac{\|\mathbf{x} - \hat{\mathbf{x}}\|}{\|\mathbf{x}\|} \leq K(A) \frac{\|\boldsymbol{\delta}\mathbf{b}\|}{\|\mathbf{b}\|}. \tag{21}$$

In order to apply this result to the present problem, assume that $A\mathbf{x} = \mathbf{b}$ are the normal equations in matrix form arising from a least squares fit for the amplitudes and phases of several tidal constituents. In particular, assume that the right-hand sides, $\mathbf{b}$ and $\mathbf{b}+\boldsymbol{\delta}\mathbf{b}$ respectively, are calculated from observations without and with background noise, i.e. $\mathbf{b}$ assumes observations from a signal that is comprised purely from tidal components whereas $\mathbf{b} + \boldsymbol{\delta}\mathbf{b}$ assume the same signal plus noise. The effect of seeking amplitudes and phases corresponding to frequencies, σ_1 and σ_2, that are relatively close, i.e. $|\sigma_2 - \sigma_1| < T^{-1}$, is to make the appropriate rows in A more linearly dependent (see Foreman, 1977, p. 19, regarding the structure of A), and so increase $K(A)$. Hence in the presence of substantial background noise, one can expect a significant difference between the calculated set of parameters, $\hat{\mathbf{x}}$, and their *true* values, $\mathbf{x}$. Assuming that accurate inference parameters are available, inference in such a case would yield better results because solving for the parameters of only one frequency, σ_1 or σ_2, would remove the linearly dependent rows and reduce $K(A)$. On the other hand, if the noise level is very small, the effect of a large condition number resulting from two close frequencies would be counteracted and a reasonably accurate set of parameters, $\hat{\mathbf{x}}$, could be expected without inference.

Table 2 gives the results of tests designed to demonstrate these points. Two 15-day records of hourly tidal heights were simulated, one using only the constituents, Z_0, O_1, K_1, M_2 and S_2, and the other with these same constituents plus random background noise. Specifically, the tidal signal varied over the range [2.77, 47.23] while the uniformly distributed random noise added the range $[-2.5,\ 2.5]$. Three sets of six consecutive 60-h harmonic analyses were executed: the first searching directly for all constituents, the second searching for only Z_0, K_1 and M_2, and the third one extending the second by inferring O_1 and S_2 from K_1 and M_2 respectively (inference parameters were calculated from a 15-day analysis which sought all constituents). In order to compare performances, means and standard deviations were calculated for each amplitude and phase over the six analyses in each series.

Results from the analyses of the tidal record with no background noise (Tests 1 to 4, Table 2) demonstrate a clear advantage to seeking all constituents directly in the least squares fit. The small non-zero standard deviations are attributable wholly to the fact that the data were rounded to four digits, making $\|\boldsymbol{\delta}\mathbf{b}\|$ slightly larger than zero. The standard deviations for the runs with inference (Test 4) are not zero because of simplifying assumptions in the inference method itself.

However, when random noise with range $[-2.5,\ 2.5]$ is also present in the tidal record (Tests 5 to 8), the standard deviations for the inference runs (Test 8) are consistently less than those obtained by the direct inclusion of all constituents in the least squares fit. This is a consequence of a reduction in $K(A)$ from 120.1 for Test 6 to 2.884 for Test 7.[4] (Corresponding values for

[3] The conventional condition number, $K(A)$, defined here, differs from the normalized condition number, $C(A)$, calculated during the solution of the normal equations in hourly heights analysis (Foreman, 1977, p. 23). Whereas $K(A)$ is unity for a diagonal matrix and assumes higher values for more ill-conditioned matrices, $C(A)$ lies between 0 and 1; 0 corresponds to a singular matrix and 1 to a diagonal matrix.

[4] The L_∞ norm was used in equation (21).

Table 2 Test Analyses With and Without Inference. Amplitude is in Metres and Phase is in Degrees.

			Test Number	Z_0	O_1		K_1		M_2		S_2	
				Amp.	Amp.	Phase	Amp.	Phase	Amp.	Phase	Amp.	Phase
Times Series Record Comprised of Tidal Signals Only	15-Day Analysis		1	25.000	3.6927	297.5	5.2474	179.1	9.0649	220.3	4.2249	69.9
	Direct Inclusion of all Constituents	Mean	2	25.000	3.6921	297.5	5.2473	179.1	9.0640	220.3	4.2254	69.9
		St. Dev.		0.000	0.0053	0.0	0.0021	0.1	0.0033	0.0	0.0022	0.0
	Direct Inclusion of only Z_0, K_1, M_2	Mean	3	25.043	—	—	6.1607	179.4	9.5420	220.4	—	—
		St. Dev.		0.056	—	—	2.5702	32.0	3.0983	19.6	—	—
	Direct Inclusion of Z_0, K_1, M_2; Inference of O_1, S_2	Mean	4	25.043	3.7212	298.7	5.2879	180.3	9.0395	220.3	4.2130	69.8
		St. Dev.		0.056	0.1167	2.5	0.1658	2.5	0.1267	0.5	0.0590	0.5
Times Series Record Comprised of Tidal Signals and Uniform $[-2.5, 2.5]$ Random Noise	15-Day Analysis		5	24.998	3.6464	296.5	5.3706	178.4	9.0095	220.5	4.3133	70.9
	Direct Inclusion of all Constituents	Mean	6	24.982	3.6099	292.4	5.4752	178.6	9.5465	222.1	5.1590	75.5
		St. Dev.		0.171	0.7488	5.0	0.6763	7.0	1.2292	4.8	0.9437	9.7
	Direct Inclusion of only Z_0, K_1, M_2	Mean	7	25.025	—	—	6.1948	178.3	9.4681	220.2	—	—
		St. Dev.		0.150	—	—	2.6406	29.2	3.2069	19.5	—	—
	Direct Inclusion of Z_0, K_1, M_2; Inference O_1, S_2	Mean	8	25.025	3.6014	297.2	5.3043	179.1	8.9170	220.1	4.2690	70.5
		St. Dev.		0.150	0.2047	3.8	0.3016	3.8	0.2915	1.8	0.1395	1.8

$C(A)$, the normalized condition numbers routinely included in the harmonic analysis program output, were 0.9697 and 0.0069.) The average value of $\|\boldsymbol{\delta}\mathbf{b}\|/\|\mathbf{b}\|$ for the six runs of Test 6 was 0.0531 while for Test 7, it was 0.0479. Consequently, applying equation (21), we anticipate a maximum change in C_j and S_j (Section 4) of 638% in the direct inclusion case and 13.8% when inference is used (assuming accurate inference constants are available).

However, for most tidal analyses, equation (21) cannot be applied, since $\|\boldsymbol{\delta}\mathbf{b}\|/\|\mathbf{b}\|$ is unknown. Munk and Hasselman (1964) derive a more useful formula for estimating the amplitude variances of close constituents in the presence of noise. Specifically, if the underlying noise spectrum is $S(\sigma)$ and the two neighbouring constituents have frequencies σ_1 and σ_2, then the estimated variance of either amplitude is $3S(\sigma)\pi^{-2}|\sigma_2 - \sigma_1|^{-2}T^{-3}$. Applying this result to the previous test data yields expected standard deviations of 0.576 and 0.622 in the O_1/K_1 and M_2/S_2 amplitude ratios respectively. Although these values are much closer to those in Table 2 than the upper bound estimate derived from equation (21), it should be noted that they are all underestimates.

For actual sea-level observations, Munk and Bullard (1963) estimate the geophysical noise level (due mainly to atmospheric excitation) at tidal frequencies to be $S(\sigma) = 1$ cm^2/(cycle per day). Applying this value to a 30-day harmonic analysis in which both P_1 and K_1 are sought directly (these constituents require 183 days to differ by one cycle) yields an expected standard deviation of 0.613 cm in either amplitude. However, a sequence of 12-monthly analyses at Prince Rupert, in which both P_1 and K_1 (and three other constituent pairs that also require approximately six months to differ by one cycle) was sought directly, produced standard deviation estimates of 17.2 and 18.4 cm respectively, for the amplitudes. Thus, in this case, $S(\sigma) = 1$ cm^2/(cycle per day) is a gross underestimate. It is worth noting that, with accurate inference parameters, the two amplitude standard deviations, after inference, reduce to 0.54 and 1.72 cm respectively. (In each case, this is 3.3% of the amplitude.)

PART III: COMPUTER PROGRAM FOR HIGH–LOW ANALYSIS

6 GENERAL DESCRIPTION OF PROGRAM

This program analyses irregularly-sampled tidal heights observations over a specified period of time. Although these observations are normally taken at high and low water, the program can also be used when the observations are not extreme values. Amplitudes and Greenwich phase lags are calculated for all requested constituents by a least squares fit method (Section 4), coupled with nodal modulation (Section 5.1). If the record length is such that certain important constituents cannot be resolved satisfactorily by including them directly in the least squares fit, provision is made for inference of their amplitudes and phases (Section 5.3).

6.1 Routines Required

(1) **MAIN** ...... reads input data, controls all output and calls other routines.

(2) **MGS** ...... does a least squares fit (with the Modified Gram–Schmidt Algorithm) to find coefficients of the sine and cosine terms corresponding to each of the specified constituent frequencies.

(3) **VUF** ...... reads required information and calculates the nodal and astronomical argument corrections for all constituents.

(4) **INFER** ...... reads required information and calculates the amplitude and phase of requested inferred constituents, as well as adjusting the amplitude and phase of the constituents used for inference.

(5) **GDAY** ...... returns the consecutive day number from a specific origin for any given date and vice versa.

Routines **VUF**, **INFER**, and **GDAY** are identical to those of the same name in the tidal heights analysis program used for hourly observations (Foreman, 1977).

6.2 Data Input

Three files or devices are used for that input. File reference number 8 contains the tidal constituent information that is necessary for the nodal and astronomical argument calculations; file reference number 9 contains the observed tidal heights and their times; and file reference number 5 contains analysis type and tidal station information. A listing of file reference number 8, along with sample input corresponding to file reference numbers 9 and 5, are given in Appendices 8.3, 8.4 and 8.5 respectively.

I. File reference number 8 is a subset of the similarly-numbered file in Foreman (1977, Section 1.3). For most analyses, its contents should not require changing (see Section 6.4 for the circumstances under which this file might be changed). It contains the following three types of data, and is read in through entry point **OPNVUF** of subroutine **VUF**.

(i) Two cards specifying values for the astronomical arguments `SO,HO,PO,ENPO,PPO,DS,DH,DP,DNP,DPP` in the format `(5F13.10)`.

`SO`	=	mean longitude of the moon (cycles) at the reference time origin;
`HO`	=	mean longitude of the sun (cycles) at the reference time origin;
`PO`	=	mean longitude of the lunar perigee (cycles) at the reference time origin;
`ENPO`	=	negative of the mean longitude of the ascending node (cycles) at the reference time origin;
`PPO`	=	mean longitude of the solar perigee (perihelion) at the reference time origin.

`DS,DH,DP,DNP,DPP` are their respective rates of change over a 365-day period at the reference time origin.

Although these argument values are not used by the program that was revised in October 1992, in order to maintain consistency with earlier programs, they are still required as input. Polynomial approximations are now employed to more accurately evaluate the astronomical arguments and their rates of change.

(ii) At least one card for all the main tidal constituents specifying their Doodson numbers and phase shifts, along with as many cards as are necessary for the satellite constituents. The first card for each such constituent is in the format (`6X,A5,1X,6I3,F5.2,I4`) and contains the following information:

`KON`	=	constituent name;
`II,JJ,KK,LL,MM,NN`	=	the six Doodson numbers for `KON`;
`SEMI`	=	phase correction for `KON`;
`NJ`	=	number of satellite constituents.

A blank card terminates this data type.

If `NJ>0`, information on the satellite constituents follows, three satellites per card, in the format `(11X,3(3I3,F4.2,F7.4,IX,I1,1X))`. For each satellite the values read are:

`LDEL,MDEL,NDEL`	=	the last three Doodson numbers of the main constituent subtracted from the last three Doodson numbers of the satellite constituent;
`PH`	=	phase correction of the satellite constituent relative to the phase of the main constituent;
`EE`	=	amplitude ratio of the satellite tidal potential to that of the main constituent;
`IR`	=	1 if the amplitude ratio has to be multiplied by the latitude correction factor for diurnal constituents,
	=	2 if the amplitude ratio has to be multiplied by the latitude correction factor for semi-diurnal constituents,
	=	otherwise if no correction is required to the amplitude ratio.

(iii) One card specifying each of the shallow water constituents and the main constituents from which they are derived. The format is `(6X,A5,I1,2X,4(F5.2,A5,5X))` and the respective values read are:

KON = name of the shallow water constituent;
NJ = number of main constituents from which it is derived;
COEF,KONCO = combination number and name of these main constituents.

The end of these shallow water constituents is denoted by a blank card.

II. File reference number 9 contains only one type of data, the observed tidal heights and times. For convenience, the input format was chosen to be the same as the daily high–low output produced by the tidal heights *prediction* program (Foreman, 1977, p. 31). Specifically, each record has the format (2X,I5,2I3,I2,6(I3,I2,F5.1),3X,I2) and contains the following information:

ISTN = tidal station number;
ID,IM,IY = day, month and year of subsequent observations;
ITH,ITM,HT = times (in hours and minutes) and height, of up to six observations for the specified date. If there are less than six observations for a day, they are padded to that number with the values 99,99 and 99.9 for the times and heights, respectively. If there are more than six observations on a given day, as many records are included as necessary, each with the same repeated date.
IC = century of observations. If IC=0, it is reset to 19.

Missing days and/or missing observations (highs, lows) are permissible. However, it is necessary that the records be ordered according to date. Units for the heights and time zone for the times are arbitrary in the sense that the post-analysis constituent amplitudes and phases will have the same units and time zone.

III. File reference number 5 contains five types of data:

(i) One record for the variables MF,IDERV,WT in the format (2I5,F5.2).

MF = number of constituents, including the constant term, Z_0, to be included in the least squares fit;
IDERV = 1 if all observations are extreme values and it is desired to use the derivative conditions in the least squares fit,
= 0 otherwise;
WT = weight to be applied to the derivative condition when IDERV=1. A recommended value is WT=1.0.

(ii) One record for each of the MF constituents to be included in the fit. Each record contains the variables NAME and FREQ in the format (A5,2X,F13.10). NAME is the constituent name, which should be left-justified in the alphanumeric field, while FREQ is its frequency measured in cycles/h. In order that there be sufficient information available to calculate the astronomical argument and nodal corrections, all these constituents must be included in the list given in Appendix 8.2. The order in which the constituents are input is also the order in which the results are output. The constant term Z_0 must be first.

(iii) One record in the format (8I5) containing the following information on the time period of the analysis:

`ID1,IM1,IY1,ID2,IM2,IY2,IC1,IC2` = day, month, year and century for the beginning and end of the analysis period;

(`IC1` or `IC2=0` or blank defaults to 19.)

(iv) One record in the format `(I5,5A4,1X,A4,4I5)` containing the following tidal station information:

`JSTN` = tidal station number;
`NSTN(I),I=1,5)` = tidal station name;
`ITZ` = time zone in which the observations were recorded;
`LATD,LATM` = station latitude in degrees and minutes;
`LOND,LONM` = station longitude in degrees and minutes.

(v) One record for each possible inference pair. The format is `(2(4X,A5,E16.10),2F10.3)` and the respective values read (through entry point **OPNINF** of subroutine **INFER**) are:

`KONAN` and `SIGAN` = name and frequency of the analysed constituent to be used for the inference;
`KONIN` and `SIGIN` = name and frequency of the inferred constituent;
`R` = amplitude ratio of `KONIN` to `KONAN`;
`ZETA` = Greenwich phase lag of the inferred constituent subtracted from the Greenwich phase lag of the analysed constituent.

These are terminated by one blank record.

As before, constituent names should be left-justified in the alphanumeric field, frequencies are measured in cycles/h and all constituents must belong to the list in Appendix 8.2.

6.3 Output

At present, only line printer (file reference number 6) output is produced by the high–low analysis program. The first page simply echoes the requested constituents to be included in the fit, the analysis period and the tidal station information, and the heights and times of the observed heights. The second page notes whether or not the derivative conditions were used for all observations and, if so, the value of the requested weight. It also gives the following direct results from the least squares fit: Z_0 amplitude and coefficients of the cosine and sine terms of all other constituents; the largest residual value in the overdetermined system of the equations and the residual sum of squares; the standard deviation of the right-hand sides of the overdetermined system and the rms residual error. The third and final page gives the raw amplitudes and raw local phases followed by the nodally-corrected amplitudes and Greenwich phases for all constituents. If there has been inference, these values are repeated with correction, and the new residual rms value is specified.

In the event that a least squares solution cannot be found because a dependent column is encountered during the orthogonalization procedure, a message to this effect along with the suggested corrective procedure is printed. If the column dependency is borderline, a slight increase in the value of the variable `TOLER` may be sufficient to obtain the solution. However, it is generally better to remove from the least squares analysis, the constituent (or its nearest neighbour, depending on which one has the smaller expected amplitude) corresponding to this tidal coefficient parameter. Specifically, if column $2n-1$ or $2n$ is dependent, then remove constituent n or its nearest neighbour. If inference parameters are available, the amplitude and phase for this constituent can still be obtained indirectly through inference.

The final page of output produced by the sample data input found in Appendices 8.3, 8.4 and 8.5, is listed in Appendix 8.6.

6.4 Program Conversion, Storage and Dimension Guidelines

The high–low analysis program was originally tested on the UNIVAC 1106 computer installation at the Institute of Ocean Sciences, Patricia Bay. It has been subsequently revised and tested to a wide variety of platforms, including PCs and UNIX workstations. Although the program was written in basic FORTRAN, some changes may be required before it can be used on other installations. These may include:

(i) replacing all calls to routine **INPROD** in subroutine **MGS** when the FORTRAN compiler does not permit a single column of a two-dimensional array to be passed to a one-dimensional array through a subroutine call. Such changes will not be necessary when FORTRAN compilers store two-dimensional arrays by columns (and this is the standard FORTRAN convention). However, if this condition is not met, the **INPROD** calls are located in lines 102, 122, 135, 156 and 173 of subroutine **MGS** and the replacement code is specified in the comment statements preceding these `CALL` statements.

(ii) altering the variable list structure for the `ENTRY` statements **OPNINF** and **OPNVUF**, and references to them;

(iii) changing some, or all of the file reference (or device) numbers from their present values, in order to conform with local machine restrictions;

(iv) altering the input tolerance, variable `TOLER`, for the **MGS** routine. If the inner product of an orthogonalized column with itself is less than `TOLER`, the column is considered to be dependent. Typically, `TOLER` is chosen to be less than `10**(-D)` where `D` represents the number of decimal digits of accuracy available. However, if the overdetermined matrix is poorly scaled, it may be necessary to either choose a much larger value or remove the corresponding constituent from the analysis. A conservative value of `10**(-7)` is presently chosen for `TOLER`.

The program, in its present form, requires approximately 3000 and 7300 single-precision words for the storage of its instructions and arrays respectively. A large part of this is for the array Q which stores the overdetermined system of linear equations and is presently dimensioned to handle approximately 800 observations with the derivative condition and 20 constituents. If the analysis is much smaller than this and memory requirements are restrictive on a particular installation, or there is a need to economize, the program size can be cut significantly by reducing the size of this array and resetting variables `NMAXP1` and `NMAXPM` appropriately.

In the event that changes are required to the program, restrictions on the minimum dimension of all arrays and minimal values of special parameters are as follows.

Let

- MC be the total number of constituents, including Z_0 and any inferred constituents;
- NOBS be the number of tidal height observations;
- NR be the number of input records of observed tidal heights;
- MPAR be 2*MC-1;
- NEQ be NOBS*2 if all the observations are extremes and the derivative condition is to be included for each, and NOBS otherwise.

Then, in the main program, parameters MXNDAY,NMAXP1 and NMAXPM should be at least NR,MPAR+1 and NEQ+MPAR respectively; arrays FREQ,NAME,AMP,PH,AMPC and PHG should have minimum dimension MC; arrays X and Y should have minimum dimension NOBS; arrays ITH,ITM and HT should have minimum dimension 6; array P should have minimum dimension MPAR; and array Q should have the exact dimension of NMAXPM by NMAXP1. In subroutine **MGS**, arrays Q and X have variable dimensions, and should be the same as Q and P in the main program.

The dimensions of any array in subroutine **VUF** need only be changed if a new constituent is to be added to the list in Appendix 8.2. In such an event, the contents of the data file associated with file reference number 8 must also be augmented in order to permit calculation of astronomical argument and nodal corrections for this constituent. In order to understand the structure of this file and the resultant calculations, consult Foreman (1977). Restrictions on the minimal array dimensions can be found there, also, as well as in the comment statements of the subroutine itself.

In subroutine **INFER**, array KON is passed in the argument list from the main program and so need only be dimensioned 2; and arrays KONAN,SIGAN,KONIN,SIGIN,R and ZETA can presently accommodate a maximum of nine inferred constituents.

In subroutine **CDAY**, arrays NDP and NDM should have dimension 12.

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Appendix 8.1 Results of 12-Month Hourly Harmonic Analysis at Prince Rupert, British Columbia.

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ANALYSIS OF HOURLY TIDAL HEIGHTS  STN  9354      1H  1/ 1/74 TO  24H 31/12/74
NO.OBS.=  8760    NO.PTS.ANAL.=  8760    MIDPT=12H  2/ 7/74  SEPARATION =0.97
```

NO	NAME	FREQUENCY	STN	M-Y/ M-Y	A	G	AL	GL
1	Z0	0.00000000	9354	174/1274	3.8709	0.00	3.8709	0.00
2	SA	0.00011407	9354	174/1274	0.0914	16.54	0.0914	198.90
3	SSA	0.00022816	9354	174/1274	0.0382	123.30	0.0382	283.01
4	MSM	0.00130978	9354	174/1274	0.0407	168.76	0.0407	349.95
5	MM	0.00151215	9354	174/1274	0.0315	117.82	0.0315	350.20
6	MSF	0.00282193	9354	174/1274	0.0186	138.15	0.0186	191.72
7	MF	0.00305009	9354	174/1274	0.0262	115.91	0.0262	329.20
8	ALP1	0.03439657	9354	174/1274	0.0032	160.86	0.0033	203.34
9	2Q1	0.03570635	9354	174/1274	0.0084	133.22	0.0086	358.20
10	SIG1	0.03590872	9354	174/1274	0.0105	123.60	0.0105	37.27
11	Q1	0.03721850	9354	174/1274	0.0539	126.53	0.0541	222.56
12	RHO1	0.03742087	9354	174/1274	0.0124	148.52	0.0125	292.04
13	O1	0.03873065	9354	174/1274	0.3125	132.46	0.3070	99.73
14	TAU1	0.03895881	9354	174/1274	0.0030	229.33	0.0030	171.87
15	BET1	0.04004043	9354	174/1274	0.0029	117.84	0.0028	88.70
16	NO1	0.04026859	9354	174/1274	0.0262	154.05	0.0267	226.70
17	CHI1	0.04047097	9354	174/1274	0.0028	93.06	0.0028	247.61
18	PI1	0.04143851	9354	174/1274	0.0062	133.11	0.0062	320.46
19	P1	0.04155259	9354	174/1274	0.1606	135.84	0.1609	145.26
20	S1	0.04166667	9354	174/1274	0.0238	116.11	0.0166	273.39
21	K1	0.04178075	9354	174/1274	0.5144	139.48	0.5096	120.39
22	PSI1	0.04189482	9354	174/1274	0.0075	175.58	0.0074	346.72
23	PHI1	0.04200891	9354	174/1274	0.0090	105.67	0.0089	255.08
24	THE1	0.04309053	9354	174/1274	0.0048	176.54	0.0048	333.98
25	J1	0.04329290	9354	174/1274	0.0275	150.20	0.0289	355.49
26	SO1	0.04460268	9354	174/1274	0.0064	185.04	0.0063	217.89
27	OO1	0.04483084	9354	174/1274	0.0157	162.83	0.0150	318.98
28	UPS1	0.04634299	9354	174/1274	0.0015	194.40	0.0015	227.66
29	OQ2	0.07597494	9354	174/1274	0.0030	338.90	0.0035	312.55
30	EPS2	0.07617731	9354	174/1274	0.0097	12.07	0.0103	33.82
31	2N2	0.07748710	9354	174/1274	0.0437	349.99	0.0484	194.63
32	MU2	0.07768947	9354	174/1274	0.0400	8.27	0.0409	259.93
33	N2	0.07899925	9354	174/1274	0.3952	14.90	0.3982	86.62
34	NU2	0.07920162	9354	174/1274	0.0766	16.49	0.0774	139.56
35	H1	0.08039733	9354	174/1274	0.0120	33.31	0.0125	335.21
36	M2	0.08051140	9354	174/1274	1.9565	35.79	1.9731	340.04
37	H2	0.08062547	9354	174/1274	0.0214	351.10	0.0215	118.67
38	MKS2	0.08073957	9354	174/1274	0.0053	262.78	0.0051	348.69
39	LDA2	0.08182118	9354	174/1274	0.0135	26.07	0.0136	331.20
40	L2	0.08202355	9354	174/1274	0.0507	40.58	0.0514	23.73
41	T2	0.08321926	9354	174/1274	0.0366	48.08	0.0366	225.73
42	S2	0.08333334	9354	174/1274	0.6445	59.26	0.6443	59.38
43	R2	0.08344740	9354	174/1274	0.0051	210.47	0.0064	208.40
44	K2	0.08356149	9354	174/1274	0.1738	50.65	0.1658	192.44
45	MSN2	0.08484548	9354	174/1274	0.0069	178.56	0.0070	51.21
46	ETA2	0.08507364	9354	174/1274	0.0088	67.48	0.0095	67.39
47	MO3	0.11924206	9354	174/1274	0.0065	50.39	0.0064	321.91
48	M3	0.12076710	9354	174/1274	0.0208	344.65	0.0210	81.16
49	SO3	0.12206399	9354	174/1274	0.0017	198.58	0.0017	165.97
50	MK3	0.12229215	9354	174/1274	0.0035	11.60	0.0035	296.76
51	SK3	0.12511408	9354	174/1274	0.0176	139.42	0.0174	120.46
52	MN4	0.15951066	9354	174/1274	0.0031	220.93	0.0031	236.90
53	M4	0.16102280	9354	174/1274	0.0060	249.00	0.0061	137.49

54	SN4	0.16233259	9354	174/1274	0.0012	262.95	0.0012	334.79
55	MS4	0.16384473	9354	174/1274	0.0023	267.54	0.0023	211.90
56	MK4	0.16407290	9354	174/1274	0.0018	210.31	0.0018	296.34
57	S4	0.16666667	9354	174/1274	0.0021	247.44	0.0021	247.69
58	SK4	0.16689484	9354	174/1274	0.0019	246.98	0.0018	28.89
59	2MK5	0.20280355	9354	174/1274	0.0043	231.99	0.0044	101.39
60	2SK5	0.20844743	9354	174/1274	0.0004	167.91	0.0004	149.07
61	2MN6	0.24002205	9354	174/1274	0.0019	152.24	0.0020	112.45
62	M6	0.24153420	9354	174/1274	0.0032	173.70	0.0033	6.44
63	2MS6	0.24435614	9354	174/1274	0.0027	199.91	0.0027	88.53
64	2MK6	0.24458429	9354	174/1274	0.0011	185.3	0.0011	215.61
65	2SM6	0.24717808	9354	174/1274	0.0004	181.56	0.0004	126.05
66	MSK6	0.24740623	9354	174/1274	0.0005	213.46	0.0004	299.61
67	3MK7	0.28331494	9354	174/1274	0.0018	294.07	0.0018	107.73
68	M8	0.32204559	9354	174/1274	0.0027	224.73	0.0028	1.71

Appendix 8.2 List of Possible Constituents (and Their Frequencies) in the High–Low Computer Program Analysis.

Z0	0.0	SA	0.0001140741
SSA	0.0002281591	MSM	0.0013097808
MM	0.0015121518	MSF	0.0028219327
MF	0.0030500918	ALP1	0.0343965699
2Q1	0.0357063507	SIG1	0.0359087218
Q1	0.0372185026	RHO1	0.0374208736
O1	0.0387306544	TAU1	0.0389588136
BET1	0.0400404353	NO1	0.0402685944
CHI1	0.0404709654	PI1	0.0414385130
P1	0.0415525871	S1	0.0416666721
K1	0.0417807462	PSI1	0.0418948203
PHI1	0.0420089053	THE1	0.0430905270
J1	0.0432928981	2PO1	0.0443745198
SO1	0.0446026789	OO1	0.0448308380
UPS1	0.0463429898	ST36	0.0733553835
2NS2	0.0746651643	ST37	0.0748675353
ST1	0.0748933234	OQ2	0.0759749451
EPS2	0.0761773161	ST2	0.0764054753
ST3	0.0772331498	O2	0.0774613089
2N2	0.0774870970	MU2	0.0776894680
SNK2	0.0787710897	N2	0.0789992488
NU2	0.0792016198	ST4	0.0794555670
OP2	0.0802832416	GAM2	0.0803090296
H1	0.0803973266	M2	0.0805114007
H2	0.0806254748	MKS2	0.0807395598
ST5	0.0809677189	ST6	0.0815930224
LDA2	0.0818211815	L2	0.0820235525
2SK2	0.0831051742	T2	0.0832192592
S2	0.0833333333	R2	0.0834474074
K2	0.0835614924	MSN2	0.0848454852
ETA2	0.0850736443	ST7	0.0853018034
2SM2	0.0861552660	ST38	0.0863576370
SKM2	0.0863834251	2SN2	0.0876674179
NO3	0.1177299033	MO3	0.1192420551
M3	0.1207671010	NK3	0.1207799950
SO3	0.1220639878	MK3	0.1222921469
SP3	0.1248859204	SK3	0.1251140796
ST8	0.1566887168	N4	0.1579984976
3MS4	0.1582008687	ST39	0.1592824904
MN4	0.1595106495	ST9	0.1597388086
ST40	0.1607946422	M4	0.1610228013
ST10	0.1612509604	SN4	0.1623325821
KN4	0.1625607413	MS4	0.1638447340
MK4	0.1640728931	SL4	0.1653568858
S4	0.1666666667	SK4	0.1668948258
MNO5	0.1982413039	2MO5	0.1997534558
3MP5	0.1999816149	MNK5	0.2012913957
2MP5	0.2025753884	2MK5	0.2028035475
MSK5	0.2056254802	3KM5	0.2058536393
2SK5	0.2084474129	ST11	0.2372259056
2NM6	0.2385098983	ST12	0.2387380574
2MN6	0.2400220501	ST13	0.2402502093
ST41	0.2413060429	M6	0.2415342020

MSN6	0.2428439828	MKN6	0.2430721419
ST42	0.2441279756	2MS6	0.2443561347
2MK6	0.2445842938	NSK6	0.2458940746
2SM6	0.2471780673	MSK6	0.2474062264
S6	0.2500000000	ST14	0.2787527046
ST15	0.2802906445	M7	0.2817899023
ST16	0.2830867891	3MK7	0.2833149482
ST17	0.2861368809	ST18	0.3190212990
3MN8	0.3205334508	ST19	0.3207616099
M8	0.3220456027	ST20	0.3233553835
ST21	0.3235835426	3MS8	0.3248675353
3MK8	0.3250956944	ST22	0.3264054753
ST23	0.3276894680	ST24	0.3279176271
ST25	0.3608020452	ST26	0.3623141970
4MK9	0.3638263489	ST27	0.3666482815
ST28	0.4010448515	M10	0.4025570033
ST29	0.4038667841	ST30	0.4053789360
ST31	0.4069168759	ST32	0.4082008687
ST33	0.4471596822	M12	0.4830684040
ST34	0.4858903367	ST35	0.4874282766

Appendix 8.3 Data Input on File Reference Number 8 for the Computer Program.

```
  .7428797055  .7771900329  .5187051308  .3631582592  .7847990160  000GMT 1/1/76
13.3594019864  .9993368945  .1129517942  .0536893056  .0000477414  INCR./365DAYS
      Z0       0  0  0  0  0  0 0.0    0
      SA       0  0  1  0  0 -1 0.0    0
      SSA      0  0  2  0  0  0 0.0    0
      MSM      0  1 -2  1  0  0  .00   0
      MM       0  1  0 -1  0  0 0.0    0
      MSF      0  2 -2  0  0  0 0.0    0
      MF       0  2  0  0  0  0 0.0    0
      ALP1     1 -4  2  1  0  0 -.25   2
      ALP1  -1  0  0 .75 0.0360R1   0 -1  0 .00 0.1906
      2Q1     1 -3  0  2  0  0-0.25   5
      2Q1   -2 -2  0 .50 0.0063    -1 -1  0 .75 0.0241R1  -1  0  0 .75 0.0607R1
      2Q1    0 -2  0 .50 0.0063     0 -1  0 .0  0.1885
      SIG1    1 -3  2  0  0  0-0.25   4
      SIG1  -1  0  0 .75 0.0095R1   0 -2  0 .50 0.0061     0 -1  0 .0  0.1884
      SIG1   2  0  0 .50 0.0087
      Q1      1 -2  0  1  0  0-0.25  10
      Q1    -2 -3  0 .50 0.0007    -2 -2  0 .50 0.0039    -1 -2  0 .75 0.0010R1
      Q1    -1 -1  0 .75 0.0115R1  -1  0  0 .75 0.0292R1   0 -2  0 .50 0.0057
      Q1    -1  0  1 .0  0.0008     0 -1  0 .0  0.1884     1  0  0 .75 0.0018R1
      Q1     2  0  0 .50 0.0028
      RHO1    1 -2  2 -1  0  0-0.25   5
      RHO1   0 -2  0 .50 0.0058     0 -1  0 .0  0.1882     1  0  0 .75 0.0131R1
      RHO1   2  0  0 .50 0.0576     2  1  0 .0  0.0175
      O1      1 -1  0  0  0  0-0.25   8
      O1    -1  0  0 .25 0.0003R1   0 -2  0 .50 0.0058     0 -1  0 .0  0.1885
      O1     1 -1  0 .25 0.0004R1   1  0  0 .75 0.0029R1   1  1  0 .25 0.0004R1
      O1     2  0  0 .50 0.0064     2  1  0 .50 0.0010
      TAU1    1 -1  2  0  0  0-0.75   5
      TAU1  -2  0  0 .0  0.0446    -1  0  0 .25 0.0426R1   0 -1  0 .50 0.0284
      TAU1   0  1  0 .50 0.2170     0  2  0 .50 0.0142
      BET1    1  0 -2  1  0  0 -.75   1
      BET1   0 -1  0 .00 0.2266
      NO1     1  0  0  1  0  0-0.75   9
      NO1   -2 -2  0 .50 0.0057    -2 -1  0 .0  0.0665    -2  0  0 .0  0.3596
      NO1   -1 -1  0 .75 0.0331R1  -1  0  0 .25 0.2227R1  -1  1  0 .75 0.0290R1
      NO1    0 -1  0 .50 0.0290     0  1  0 .0  0.2004     0  2  0 .50 0.0054
      CHI1    1  0  2 -1  0  0-0.75   2
      CHI1   0 -1  0 .50 0.0282     0  1  0 .0  0.2187
      PI1     1  1 -3  0  0  1-0.25   1
      PI1    0 -1  0 .50 0.0078
      P1      1  1 -2  0  0  0-0.25   6
      P1     0 -2  0 .0  0.0008     0 -1  0 .50 0.0112     0  0  2 .50 0.0004
      P1     1  0  0 .75 0.0004R1   2  0  0 .50 0.0015     2  1  0 .50 0.0003
      S1      1  1 -1  0  0  1-0.75   2
      S1     0  0 -2 .0  0.3534     0  1  0 .50 0.0264
      K1      1  1  0  0  0  0-0.75  10
      K1    -2 -1  0 .0  0.0002    -1 -1  0 .75 0.0001R1  -1  0  0 .25 0.0007R1
      K1    -1  1  0 .75 0.0001R1   0 -2  0 .0  0.0001     0 -1  0 .50 0.0198
      K1     0  1  0 .0  0.1356     0  2  0 .50 0.0029     1  0  0 .25 0.0002R1
      K1     1  1  0 .25 0.0001R1
      PSI1    1  1  1  0  0 -1-0.75   1
      PSI1   0  1  0 .0  0.0190
```

```
PHI1     1  1  2  0  0  0-0.75   5
PHI1   -2  0  0 .0  0.0344    -2  1  0 .0  0.0106      0  0 -2 .0  0.0132
PHI1    0  1  0 .50 0.0384     0  2  0 .50 0.0185
THE1     1  2 -2  1  0  0 -.75   4
THE1   -2 -1  0 .00  .0300    -1  0  0 .25 0.0141R1    0 -1  0 .50  .0317
THE1    0  1  0 .00  .1993
J1       1  2  0 -1  0  0-0.75  10
J1      0 -1  0 .50 0.0294     0  1  0 .0  0.1980      0  2  0 .50 0.0047
J1      1 -1  0 .75 0.0027R1   1  0  0 .25 0.0816R1    1  1  0 .25 0.0331R1
J1      1  2  0 .25 0.0027R1   2  0  0 .50 0.0152      2  1  0 .50 0.0098
J1      2  2  0 .50 0.0057
OO1      1  3  0  0  0  0-0.75   8
OO1    -2 -1  0 .50 0.0037    -2  0  0 .0  0.1496     -2  1  0 .0  0.0296
OO1    -1  0  0 .25 0.0240R1  -1  1  0 .25 0.0099R1    0  1  0 .0  0.6398
OO1     0  2  0 .0  0.1342     0  3  0 .0  0.0086
UPS1     1  4  0 -1  0  0 -.75   5
UPS1   -2  0  0 .00 0.0611     0  1  0 .00 0.6399      0  2  0 .00 0.1318
UPS1    1  0  0 .25 0.0289R1   1  1  0 .25 0.0257R1
OQ2      2 -3  0  3  0  0 0.0    2
OQ2    -1  0  0 .25 0.1042R2   0 -1  0 .50 0.0386
EPS2     2 -3  2  1  0  0 0.0    3
EPS2   -1 -1  0 .25 0.0075R2  -1  0  0 .25 0.0402R2    0 -1  0 .50 0.0373
2N2      2 -2  0  2  0  0 0.0    4
2N2    -2 -2  0 .50 0.0061    -1 -1  0 .25 0.0117R2   -1  0  0 .25 0.0678R2
2N2     0 -1  0 .50 0.0374
MU2      2 -2  2  0  0  0 0.0    3
MU2    -1 -1  0 .25 0.0018R2  -1  0  0 .25 0.0104R2    0 -1  0 .50 0.0375
N2       2 -1  0  1  0  0 0.0    4
N2     -2 -2  0 .50 0.0039    -1  0  1 .00 0.0008      0 -2  0 .00 0.0005
N2      0 -1  0 .50 0.0373
NU2      2 -1  2 -1  0  0 0.0    4
NU2     0 -1  0 .50 0.0373     1  0  0 .75 0.0042R2    2  0  0 .0  0.0042
NU2     2  1  0 .50 0.0036
GAM2     2  0 -2  2  0  0 -.50   3
GAM2   -2 -2  0 .00 0.1429    -1  0  0 .25 0.0293R2    0 -1  0 .50 0.0330
H1       2  0 -1  0  0  1-0.50   2
H1      0 -1  0 .50 0.0224     1  0 -1 .50 0.0447
M2       2  0  0  0  0  0 0.0    9
M2     -1 -1  0 .75 0.0001R2  -1  0  0 .75 0.0004R2    0 -2  0 .0  0.0005
M2      0 -1  0 .50 0.0373     1 -1  0 .25 0.0001R2    1  0  0 .75 0.0009R2
M2      1  1  0 .75 0.0002R2   2  0  0 .0  0.0006      2  1  0 .0  0.0002
H2       2  0  1  0  0 -1 0.0    1
H2      0 -1  0 .50 0.0217
LDA2     2  1 -2  1  0  0-0.50   1
LDA2    0 -1  0 .50 0.0448
L2       2  1  0 -1  0  0-0.50   5
L2      0 -1  0 .50 0.0366     2 -1  0 .00 0.0047      2  0  0 .50 0.2505
L2      2  1  0 .50 0.1102     2  2  0 .50 0.0156
T2       2  2 -3  0  0  1 0.0    0
S2       2  2 -2  0  0  0 0.0    3
S2      0 -1  0 .0  0.0022     1  0  0 .75 0.0001R2    2  0  0 .0  0.0001
R2       2  2 -1  0  0 -1-0.50   2
R2      0  0  2 .50 0.2535     0  1  2 .0  0.0141
K2       2  2  0  0  0  0 0.0    5
K2     -1  0  0 .75 0.0024R2  -1  1  0 .75 0.0004R2    0 -1  0 .50 0.0128
K2      0  1  0 .0  0.2980     0  2  0 .0  0.0324
```

```
ETA2     2  3  0 -1  0  0 0.0     7
ETA2    0 -1  0 .50 0.0187     0  1  0 .0  0.4355     0  2  0 .0  0.0467
ETA2    1  0  0 .75 0.0747R2   1  1  0 .75 0.0482R2   1  2  0 .75 0.0093R2
ETA2    2  0  0 .50 0.0078
M3       3  0  0  0  0  0 -.50   1
M3      0 -1  0 .50  .0564

2PO1 2   2.0 P1         -1.0 O1
SO1  2   1.0 S2         -1.0 O1
ST36 3   2.0 M2          1.0 N2         -2.0 S2
2NS2 2   2.0 N2         -1.0 S2
ST37 2   3.0 M2         -2.0 S2
ST1  3   2.0 N2          1.0 K2         -2.0 S2
ST2  4   1.0 M2          1.0 N2          1.0 K2         -2.0 S2
ST3  3   2.0 M2          1.0 S2         -2.0 K2
O2   1   2.0 O1
ST4  3   2.0 K2          1.0 N2         -2.0 S2
SNK2 3   1.0 S2          1.0 N2         -1.0 K2
OP2  2   1.0 O1          1.0 P1
MKS2 3   1.0 M2          1.0 K2         -1.0 S2
ST5  3   1.0 M2          2.0 K2         -2.0 S2
ST6  4   2.0 S2          1.0 N2         -1.0 M2         -1.0 K2
2SK2 2   2.0 S2         -1.0 K2
MSN2 3   1.0 M2          1.0 S2         -1.0 N2
ST7  4   2.0 K2          1.0 M2         -1.0 S2         -1.0 N2
2SM2 2   2.0 S2         -1.0 M2
ST38 3   2.0 M2          1.0 S2         -2.0 N2
SKM2 3   1.0 S2          1.0 K2         -1.0 M2
2SN2 2   2.0 S2         -1.0 N2
NO3  2   1.0 N2          1.0 O1
MO3  2   1.0 M2          1.0 O1
NK3  2   1.0 N2          1.0 K1
SO3  2   1.0 S2          1.0 O1
MK3  2   1.0 M2          1.0 K1
SP3  2   1.0 S2          1.0 P1
SK3  2   1.0 S2          1.0 K1
ST8  3   2.0 M2          1.0 N2         -1.0 S2
N4   1   2.0 N2
3MS4 2   3.0 M2         -1.0 S2
ST39 4   1.0 M2          1.0 S2          1.0 N2         -1.0 K2
MN4  2   1.0 M2          1.0 N2
ST40 3   2.0 M2          1.0 S2         -1.0 K2
ST9  4   1.0 M2          1.0 N2          1.0 K2         -1.0 S2
M4   1   2.0 M2
ST10 3   2.0 M2          1.0 K2         -1.0 S2
SN4  2   1.0 S2          1.0 N2
KN4  2   1.0 K2          1.0 N2
MS4  2   1.0 M2          1.0 S2
MK4  2   1.0 M2          1.0 K2
SL4  2   1.0 S2          1.0 L2
S4   1   2.0 S2
SK4  2   1.0 S2          1.0 K2
MNO5 3   1.0 M2          1.0 N2          1.0 O1
2MO5 2   2.0 M2          1.0 O1
3MP5 2   3.0 M2         -1.0 P1
MNK5 3   1.0 M2          1.0 N2          1.0 K1
2MP5 2   2.0 M2          1.0 P1
```

2MK5	2	2.0 M2	1.0 K1		
MSK5	3	1.0 M2	1.0 S2	1.0 K1	
3KM5	3	1.0 K2	1.0 K1	1.0 M2	
2SK5	2	2.0 S2	1.0 K1		
ST11	3	3.0 N2	1.0 K2	-1.0 S2	
2NM6	2	2.0 N2	1.0 M2		
ST12	4	2.0 N2	1.0 M2	1.0 K2	-1.0 S2
ST41	3	3.0 M2	1.0 S2	-1.0 K2	
2MN6	2	2.0 M2	1.0 N2		
ST13	4	2.0 M2	1.0 N2	1.0 K2	-1.0 S2
M6	1	3.0 M2			
MSN6	3	1.0 M2	1.0 S2	1.0 N2	
MKN6	3	1.0 M2	1.0 K2	1.0 N2	
2MS6	2	2.0 M2	1.0 S2		
2MK6	2	2.0 M2	1.0 K2		
NSK6	3	1.0 N2	1.0 S2	1.0 K2	
2SM6	2	2.0 S2	1.0 M2		
MSK6	3	1.0 M2	1.0 S2	1.0 K2	
ST42	3	2.0 M2	2.0 S2	-1.0 K2	
S6	1	3.0 S2			
ST14	3	2.0 M2	1.0 N2	1.0 O1	
ST15	3	2.0 N2	1.0 M2	1.0 K1	
M7	1	3.5 M2			
ST16	3	2.0 M2	1.0 S2	1.0 O1	
3MK7	2	3.0 M2	1.0 K1		
ST17	4	1.0 M2	1.0 S2	1.0 K2	1.0 O1
ST18	2	2.0 M2	2.0 N2		
3MN8	2	3.0 M2	1.0 N2		
ST19	4	3.0 M2	1.0 N2	1.0 K2	-1.0 S2
M8	1	4.0 M2			
ST20	3	2.0 M2	1.0 S2	1.0 N2	
ST21	3	2.0 M2	1.0 N2	1.0 K2	
3MS8	2	3.0 M2	1.0 S2		
3MK8	2	3.0 M2	1.0 K2		
ST22	4	1.0 M2	1.0 S2	1.0 N2	1.0 K2
ST23	2	2.0 M2	2.0 S2		
ST24	3	2.0 M2	1.0 S2	1.0 K2	
ST25	3	2.0 M2	2.0 N2	1.0 K1	
ST26	3	3.0 M2	1.0 N2	1.0 K1	
4MK9	2	4.0 M2	1.0 K1		
ST27	3	3.0 M2	1.0 S2	1.0 K1	
ST28	2	4.0 M2	1.0 N2		
M10	1	5.0 M2			
ST29	3	3.0 M2	1.0 N2	1.0 S2	
ST30	2	4.0 M2	1.0 S2		
ST31	4	2.0 M2	1.0 N2	1.0 S2	1.0 K2
ST32	2	3.0 M2	2.0 S2		
ST33	3	4.0 M2	1.0 S2	1.0 K1	
M12	1	6.0 M2			
ST34	2	5.0 M2	1.0 S2		
ST35	4	3.0 M2	1.0 N2	1.0 K2	1.0 S2

Appendix 8.4 Sample Data Input on File Reference Number 9 for the Computer Program: Prince Rupert High and Low Water Observations for January 1974.

```
9354  1  174  619534.8 1252220.7 1830457.9 9999 99.9 9999 99.9 9999 99.9
9354  2  174  019211.1  720532.8 1359191.0 1944426.8 9999 99.9 9999 99.9
9354  3  174  132244.9  818549.5 15 7176.8 2123447.0 9999 99.9 9999 99.9
9354  4  174  248269.2  919580.8 1613140.0 2244482.7 9999 99.9 9999 99.9
9354  5  174  357264.9 1015616.5 1710 93.2 2335518.5 9999 99.9 9999 99.9
9354  6  174  5 1249.0 1111657.0 18 3 44.9 9999 99.9 9999 99.9 9999 99.9
9354  7  174  029557.3  557216.9 12 6680.3 1851  9.3 9999 99.9 9999 99.9
9354  8  174  118595.9  651191.2 1256704.2 1941-17.1 9999 99.9 9999 99.9
9354  9  174  2 4624.1  742159.7 1349712.3 2023-21.8 9999 99.9 9999 99.9
9354 10  174  246648.7  833138.5 1440696.4 21 8 -8.8 9999 99.9 9999 99.9
9354 11  174  328662.5  918143.1 1531677.9 2148 54.6 9999 99.9 9999 99.9
9354 12  174  411685.2 1012167.8 1617646.3 2226105.7 9999 99.9 9999 99.9
9354 13  174  453674.4 1112190.1 17 8585.6 2312153.0 9999 99.9 9999 99.9
9354 14  174  542645.0 12 3220.9 18 5551.1 2357236.9 9999 99.9 9999 99.9
9354 15  174  631638.4 1314243.7 19 9516.4 9999 99.9 9999 99.9 9999 99.9
9354 16  174  054288.8  736603.1 1438251.2 2036492.7 9999 99.9 9999 99.9
9354 17  174  210329.7  850601.6 1554235.8 2225514.0 9999 99.9 9999 99.9
9354 18  174  337366.4  947648.7 17 7229.8 2313539.2 9999 99.9 9999 99.9
9354 19  174  438335.0 1038603.6 1744171.2 9999 99.9 9999 99.9 9999 99.9
9354 20  174  0 8533.3  527292.7 1130615.4 1816144.2 9999 99.9 9999 99.9
9354 21  174  049565.8  612280.1 12 7623.3 1855108.8 9999 99.9 9999 99.9
9354 22  174  124574.9  651264.4 1240637.9 1929 94.2 9999 99.9 9999 99.9
9354 23  174  150577.7  724236.4 1320644.0 1956 90.8 9999 99.9 9999 99.9
9354 24  174  214597.6  754219.7 1353650.4 2021101.4 9999 99.9 9999 99.9
9354 25  174  243605.3  832207.1 1424627.7 2051 99.4 9999 99.9 9999 99.9
9354 26  174  315602.8  9 2193.0 15 2614.0 2115118.8 9999 99.9 9999 99.9
9354 27  174  339616.5  943197.8 1538585.6 2150146.2 9999 99.9 9999 99.9
9354 28  174  4 8611.4 1015202.3 1621556.8 2219178.4 9999 99.9 9999 99.9
9354 29  174  445606.6 11 7206.7 17 0519.0 2255207.8 9999 99.9 9999 99.9
9354 30  174  526579.2 12 4203.9 1759484.0 2337256.4 9999 99.9 9999 99.9
9354 31  174  627598.4 1311236.4 19 7478.2 9999 99.9 9999 99.9 9999 99.9
```

Appendix 8.5 Sample Data Input on File Reference Number 5 for the Computer Program: A High–Low Analysis of the Observations in Appendix 8.4 which Includes the Constituents Z_0, M_m, M_{sf}, O_1, K_1, N_2, M_2 and S_2; Infers P_1, ν_2 and K_2; and Uses the Zero Derivative Information with Weighting Coefficient Equal to 1.0.

```
     8    1 1.0
 Z0      0.000
 MM      0.0015121518
 MSF     0.0028219327
 O1      0.0387306544
 K1      0.0417807462
 N2      0.0789992488
 M2      0.0805114007
 S2      0.0833333333
    01   01   74   31   01   74
  9354PRINCE RUPERT BC      PST    54   19  130   20
     K1       0.0417807462    P1       0.0415525871    0.3122    3.65
     S2       0.0833333333    K2       0.0835614924    0.2696    8.61
     N2       0.0789992488    NU2      0.0792016198    0.1938   -1.63
```

Appendix 8.6 Final Page of Computer Output Corresponding to the Input of Appendices 8.3, 8.4 and 8.5.

HARMONIC TIDAL ANALYSIS RESULTS:
RAW AMPLITUDES AND RAW LOCAL PHASES ARE FOLLOWED BY NODALLY CORRECTED AMPLITUDES AND GREENWICH PHASES

Z0	0.000000000	388.728	0.00	388.728	0.00
MM	0.001512152	22.970	23.09	22.970	128.86
MSF	0.002821933	7.844	-8.81	7.844	185.90
O1	0.038730655	31.793	-178.81	31.330	130.42
K1	0.041780747	64.604	-61.71	64.050	152.74
N2	0.078999251	41.696	-49.64	41.763	12.17
M2	0.080511399	195.485	-131.73	194.971	35.75
S2	0.083333336	68.772	71.70	68.771	71.58

THE SAME RESULTS WITH INFERENCE

Z0	0.000000000	388.728	0.00	388.728	0.00
MM	0.001512152	22.970	23.09	22.970	128.86
MSF	0.002821933	7.844	-8.81	7.844	185.90
O1	0.038730655	31.793	-178.81	31.330	130.42
P1	0.041552588	16.791	342.12	16.777	137.29
K1	0.041780751	54.201	-73.51	53.737	140.94
N2	0.078999251	35.263	-47.25	35.319	14.56
NU2	0.079201616	6.883	297.55	6.845	16.19
M2	0.080511406	195.485	-131.73	194.971	35.75
S2	0.083333336	70.285	56.87	70.284	56.75
K2	0.083561502	18.892	159.07	18.949	48.14

AND THE ROOT MEAN SQUARE RESIDUAL ERROR AFTER INFERENCE IS 0.100923E+02